UNION OF SUBSPACES SIGNAL DETECTION IN SUBSPACE INTERFERENCE

Muhammad Asad Lodhi and Waheed U. Bajwa

Department of Electrical and Computer Engineering, Rutgers University

ABSTRACT

This paper investigates detection theory for signals belonging to a union of subspaces (UoS) in the presence of an interference subspace and white noise of unknown variance. Generalized likelihood ratio tests are provided for both signal detection and "active" subspace detection under the UoS model. The paper also derives performance bounds on the associated detection problems and relates them to the geometry of subspaces in the union and the interfering subspace. These relations are then corroborated through numerical experiments on synthetic data.

***Index Terms*—** interference, signal detection, subspace detection, subspace geometry, union of subspaces

1. INTRODUCTION

One of the classic problems in signal processing is signal detection, which has been extensively explored under the subspace model [1]. Recently however, the focus has shifted towards a nonlinear generalization of the subspace model. This nonlinear model, called the *union-of-subspaces* (UoS) model [2], has proven to be a better model for real-world data [3–5]. The focus of this paper is on detecting signals that conform to the UoS model in the presence of a signal from an interfering subspace and white noise of unknown variance. To this end, we provide statistical tests and derive performance measures for detecting the signal and the associated subspace from which the signal is coming. Furthermore, we also examine the effect that the geometry of the signal and the interference subspaces has on these performance measures.

Prior work and our contributions: In terms of prior work, the problem of signal detection under the subspace model has been analyzed in the literature in great detail and under various settings [6–9]. The earliest and most well-known method is the matched subspace detector [6], which is an energy detector, i.e., it thresholds the energy in the observed signal after projecting it onto the subspace in question.

As for the UoS model, there have been several works recently that either directly investigate detection under the UoS model or examine a related problem [10–15]. For instance, [10] examines signal detection under the compressive sensing framework [16] and analyzes the resulting generalized likelihood ratio test (GLRT). Similarly, [11] evaluates the same problem but in the context of multi-target radar based detection. The work in [12] extends [10] to more general settings but is still restricted to the sparsity framework. All of these works can be considered special instances of detection under the UoS model, but they lack the interpretation of their results in terms of the geometry between the subspaces. The works most closely related to our paper are [17] and [18]. In [17], signal and active subspace detection under the UoS are studied specifically under the framework of radar target detection with a focus on target spectral signatures. This hinders a general analysis in terms of the geometry of subspaces. On the other hand, [18] investigates the recovery of a signal and detection of the corresponding subspace within the UoS through a linear sampling operator, but this work is essentially concerned with the properties of the sampling operator. Finally, none of the aforementioned works consider detection under UoS model in the presence of an interference subspace with noisy observations.

In this paper we derive GLRTs for signal and active subspace detection problems and bounds on the associated probabilities of signal detection, subspace classification and false alarm. Moreover, we also analyze these probability bounds in light of the geometry of the subspaces and characterize the effect of geometry on the overall performance. Our analysis highlights the following key observations. As the subspaces in the union move away from the interference subspace, the detection performance improves. Furthermore, as the angles between the *interference-free subspaces* (defined in Sec. 3) increase, the probability of correct classification increases. We also point out here that this paper is closely related to another paper by the authors [19], where detection problems under UoS are considered under varying colored noise conditions but without interference.

Notation and organization: Bold lowercase and bold uppercase letters are used to denote vectors and matrices, respectively. For a matrix $\mathbf{A}$, the k-th column and the (j,k)-th entry are, respectively, denoted by $\mathbf{a}_k$ and a_{jk}. Furthermore, $\mathbf{A}^{-1}$ and $|\mathbf{A}|$ are used to denote the inverse (if it exists) and the determinant of $\mathbf{A}$, respectively. For a vector $\mathbf{a}$, $\|\mathbf{a}\|_p$ and $|\mathbf{a}|$ denote its ℓ_p-norm and its elementwise absolute values, respectively. Finally, $Q(\cdot)$, $\Gamma(\cdot)$, and $K_n(\cdot)$ denote the Gaussian Q function, the Gamma function, and the modified Bes-

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sel function of the second kind with parameter n, respectively.

The rest of the paper is organized as follows. In Sec. 2, we formulate the two detection problems under the UoS model in the presence of an interference signal. Sec. 3 presents the GLRTs for these two problems and derives bounds on various performance measures. Sec. 4 provides a discussion and interpretation of the results obtained in Sec. 3. In Sec. 5, results of numerical experiments are presented, while we conclude the paper in Sec. 6.

2. PROBLEM FORMULATION

In this section we mathematically pose the two detection problems under the UoS model in the presence of an interference signal coming from a subspace, namely, *signal detection* and *active subspace detection*.

Signal detection in this regard refers to the problem of deciding whether the observed signal $\mathbf{y} \in \mathbb{R}^m$ is an unknown noisy interference signal or an unknown noisy signal of interest in the presence of an unknown interference signal. Mathematically, this problem can be expressed as the following binary hypothesis test:

$$\begin{aligned} \mathcal{H}_0: &\quad \mathbf{y} = \mathbf{t} + \mathbf{n}; \\ \mathcal{H}_1: &\quad \mathbf{y} = \mathbf{x} + \mathbf{t} + \mathbf{n}; \end{aligned} \tag{1}$$

where $\mathbf{t} \in \mathbb{R}^m$ is the unknown interference coming from an n-dimensional interference subspace T, $\mathbf{n} \in \mathbb{R}^m$ is the observation noise assumed to be white Gaussian with unknown variance σ^2, and $\mathbf{x}$ is the signal of interest belonging to a union of subspaces, i.e., $\mathbf{x} \in \bigcup_{k=1}^{K_0} S_k$, where all subspaces S_k are n-dimensional in $\mathbb{R}^m$, and $S_k \cap S_j = \emptyset$ for $k \neq j$.

The active subspace detection problem aims at identifying the subspace S_k from the union that generates the signal of interest, in addition to identifying if the signal is present. This multiple hypothesis testing problem can mathematically be posed as:

$$\begin{aligned} \mathcal{H}_0: &\quad \mathbf{y} = \mathbf{t} + \mathbf{n}; \\ \mathcal{H}_k: &\quad \mathbf{y} = \mathbf{x} + \mathbf{t} + \mathbf{n};\ \mathbf{x} \in S_k,\ k = 1, 2, \dots, K_0, \end{aligned} \tag{2}$$

where $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{x}$ are as defined before. For both of these problems, we derive tests for detection and relate the performance of these tests with the geometry of the subspaces in the union and the interference subspace. This is achieved through the use of principal angles between the subspaces, where the i-th principal angle between two subspaces S_j and S_k denoted by $\varphi_i^{(j,k)}$ for $i = 1, 2, \dots, n$, is defined as [20]: $\varphi_i^{(j,k)} = \arccos\left(\max_{\mathbf{u},\mathbf{v}}\left\{\frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|_2\|\mathbf{v}\|_2} : \mathbf{u} \in S_j, \mathbf{v} \in S_k, \mathbf{u} \perp \mathbf{u}_\ell, \mathbf{v} \perp \mathbf{v}_\ell, \ell = 1, \dots, i-1\right\}\right)$, where $(\mathbf{u}_\ell, \mathbf{v}_\ell) \in S_j \times S_k$ denote the principal vectors associated with the ℓ-th principal angle.

The performance of our tests for the aforementioned problems is described in terms of the probabilities of detection (P_D), classification (P_C), and false alarm (P_{FA}). For this purpose, let us define $P_{\mathcal{H}_i}(\cdot) = \Pr(\cdot|\mathcal{H}_i)$ and the event $\widehat{\mathcal{H}}_i = \{\text{Hypothesis } \mathcal{H}_i \text{ is accepted}\}$. Then, we have the following performance measures for the signal detection problem: $P_D = P_{\mathcal{H}_1}(\widehat{\mathcal{H}}_1)$ and $P_{FA} = P_{\mathcal{H}_0}(\widehat{\mathcal{H}}_1)$, and the following for active subspace detection problem: $P_C = \sum_{k=1}^{K_0} P_{\mathcal{H}_k}(\widehat{\mathcal{H}}_k)\Pr(\mathcal{H}_k)$ and $P_{FA} = P_{\mathcal{H}_0}(\cup_{k=1}^{K_0}\widehat{\mathcal{H}}_k)$.

For the rest of the paper we use $P_{S_k}(\cdot) = \Pr(\cdot|\{\mathbf{x} \in S_k\})$ and $\Psi(\eta_0, \alpha) = \frac{\sqrt{2}}{2^n\Gamma(n/2)}(\eta_0\alpha)^{(n-1)/2}K_{(n-1)/2}\left(\frac{\eta_0\alpha}{2}\right)$, where $\alpha \in \mathbb{R}_+$ and $\eta_0 \in (0, 1/2)$. Using this notation, we can express P_D as $P_D = \sum_{k=1}^{K_0} P_{S_k}(\widehat{\mathcal{H}}_1)\Pr(\mathbf{x} \in S_k)$.

3. STATISTICAL TESTS FOR DETECTION

Let us begin with a basis for the k-th subspace S_k, denoted by $\mathbf{H}_k \in \mathbb{R}^{m\times n}$; then a signal $\mathbf{x}$ belonging to S_k can be expressed as $\mathbf{x} = \mathbf{H}_k\boldsymbol{\theta}_k$. Similarly, an interference signal $\mathbf{t}$ belonging to the interference subspace T can be denoted by $\mathbf{t} = \mathbf{V}\boldsymbol{\mu}$, where $\mathbf{V} \in \mathbb{R}^{m\times n}$ represents a basis for the interference subspace. One can now trivially see that $\mathbf{y}|\mathcal{H}_0 \sim \mathcal{N}(\mathbf{t}, \sigma^2\mathbf{I})$ for both detection problems, $\mathbf{y}|\mathcal{H}_1 \sim \mathcal{N}(\mathbf{x} + \mathbf{t}, \sigma^2\mathbf{I})$ for signal detection, and $\mathbf{y}|\mathcal{H}_k \sim \mathcal{N}(\mathbf{H}_k\boldsymbol{\theta}_k + \mathbf{V}\boldsymbol{\mu}, \sigma^2\mathbf{I})$ for active subspace detection. Since the quantities $\mathbf{t}$, $\mathbf{x}$, $\boldsymbol{\mu}$, and $\boldsymbol{\theta}_k$ are unknown, we resort to the use of GLRTs instead of simple likelihood ratio tests. Our test for signal detection is presented in the theorem below and uses the following definitions: let the projection matrix for the interference subspace be $\mathbf{P}_T$, the projection matrix for the subspace orthogonal to T be $\mathbf{P}_T^\perp$, the projection matrix for the k-th subspace S_k be $\mathbf{P}_{S_k}$, the projection matrix for the subspace orthogonal to both T and S_k be $\mathbf{P}_{S_kT}^\perp$, and the part of the k-th subspace S_k that is orthogonal to the interference subspace T be $\bar{S}_k$ (we will call this k-th *interference-free subspace* from here on) with its projection matrix being $\mathbf{P}_{\bar{S}_k} = \mathbf{P}_T^\perp\mathbf{H}_k(\mathbf{H}_k^T\mathbf{P}_T^\perp\mathbf{H}_k)^{-1}\mathbf{H}_k^T\mathbf{P}_T^\perp$.

Theorem 1. *Fix a test threshold $\bar{\gamma} > 0$, and define $\widehat{k} = \arg\min_k(\mathbf{y}^T\mathbf{P}_{S_kT}^\perp\mathbf{y})$ and $R_{\mathbf{y}}^T(\mathbf{P}_{S_{\widehat{k}}T}^\perp) = \frac{\mathbf{y}^T\mathbf{P}_T^\perp\mathbf{y}}{\mathbf{y}^T\mathbf{P}_{S_{\widehat{k}}T}^\perp\mathbf{y}}$. Then the GLRT for the signal detection and the active subspace detection problems under the UoS model with interference is, respectively, given by*

$$R_{\mathbf{y}}^T(\mathbf{P}_{S_{\widehat{k}}T}^\perp) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \bar{\gamma} \qquad \textit{and} \qquad R_{\mathbf{y}}^T(\mathbf{P}_{S_{\widehat{k}}T}^\perp) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_{\widehat{k}}}{\gtrless}} \bar{\gamma}\,. \tag{3}$$

Proof. The proof of this theorem follows the steps from our earlier work [19, Theorem 1]. Specifically, one can work out the following estimates under the detection problem: $\widehat{\mathbf{t}}|\mathcal{H}_0 = \mathbf{P}_T\mathbf{y}$, $\widehat{\sigma}^2|\mathcal{H}_0 = \|\mathbf{y} - \mathbf{P}_T\mathbf{y}\|_2^2/m$, $[\widehat{\mathbf{x}};\ \widehat{\mathbf{t}}]|\mathcal{H}_1 = \mathbf{P}_{S_{\widehat{k}}T}\mathbf{y}$, and

$\widehat{\sigma}^2|\mathcal{H}_1 = \|\mathbf{y} - \mathbf{P}_{S_{\widehat{k}}T}\mathbf{y}\|_2^2/m$, for $\widehat{k}$ as defined in the theorem. With these estimates, one can use the GLRT to arrive at the test statistic for signal detection. A similar procedure can be followed for active subspace detection after deriving the following estimates for the alternate hypotheses: $[\widehat{\mathbf{x}};\ \widehat{\mathbf{t}}]|\mathcal{H}_k = \mathbf{P}_{S_kT}\mathbf{y}$ and $\widehat{\sigma}^2|\mathcal{H}_k = \|\mathbf{y} - \mathbf{P}_{S_kT}\mathbf{y}\|_2^2/m$. ■

Various aspects of the two tests stated in Theorem 1 are discussed in Sec 4. The performance of the test statistics in Theorem 1 can be described by the following theorem in terms of bounds on the performance measures. We resort to bounds on these measures because of the complicated joint distributions of dependent variables that would appear in the exact expressions (see [19] for more details).

Theorem 2. *The GLRTs in Theorem 1 result in probability of false alarm that is upper bounded using:*

$$P_{FA} \leq \min\Big\{1\ ,\ \sum_{k=1}^{K_0} \Pr\Big(R_{\mathbf{y}}^T(\mathbf{P}_{S_kT}^{\perp}) > \bar{\gamma}\Big)\Big\}. \quad (4)$$

Further, for signal detection, the probability of detection $P_D = \sum_{k=1}^{K_0} P_{S_k}(\widehat{\mathcal{H}}_1)\Pr(\mathbf{x} \in S_k)$ *can be bounded using:*

$$P_{S_k}(\widehat{\mathcal{H}}_1) \geq \sum_{i=1}^{K_0} \frac{\Big[P_{S_k}\Big(R_{\mathbf{y}}^T(\mathbf{P}_{\bar{S}_iT}^{\perp}) > \bar{\gamma}\Big)\Big]^2}{\sum\limits_{j=1}^{K_0} P_{S_k}\Big(R_{\mathbf{y}}^T(\mathbf{P}_{\bar{S}_iT}^{\perp}) > \bar{\gamma}, R_{\mathbf{y}}^T(\mathbf{P}_{\bar{S}_jT}^{\perp}) > \bar{\gamma}\Big)},$$

$$\textit{and } P_{S_k}(\widehat{\mathcal{H}}_1) \leq \min\Big\{1\ ,\ \sum_{i=1}^{K_0} P_{S_k}\Big(R_{\mathbf{y}}^T(\mathbf{P}_{\bar{S}_iT}^{\perp}) > \bar{\gamma}\Big)\Big\}. \quad (5)$$

Finally, by defining $R_{\mathbf{y}}(\mathbf{P}_{\bar{S}_k}, \mathbf{P}_{\bar{S}_j}) = \frac{\mathbf{y}^T\mathbf{P}_{\bar{S}_k}\mathbf{y}}{\mathbf{y}^T\mathbf{P}_{\bar{S}_j}\mathbf{y}}$*, the probability of classification* P_C *for active subspace detection can be lower bounded as:*

$$P_{\mathcal{H}_k}(\widehat{\mathcal{H}}_k) \geq \max\Big\{0\ ,\ P_{S_k}(R_{\mathbf{y}}^T(\mathbf{P}_{\bar{S}_kT}^{\perp}) > \bar{\gamma}) + \sum_{j=1, j\neq k}^{K_0} P_{S_k}(R_{\mathbf{y}}(\mathbf{P}_{\bar{S}_k}, \mathbf{P}_{\bar{S}_j}) > 1) - (K_0 - 1)\Big\}. \quad (6)$$

The proof of Theorem 2 follows the steps from our earlier work [19, Theorem 2]. Another lower bound on (6) can be achieved using [18, Lemma 1] as $P_{\mathcal{H}_k}(\widehat{\mathcal{H}}_k) \geq \max\big\{0, P_{S_k}(R_{\mathbf{y}}^T(\mathbf{P}_{\bar{S}_iT}^{\perp}) > \bar{\gamma}) - \sum_{j:j\neq k} Q\big(\frac{1}{2}(1-2\eta_0)\sqrt{\lambda_{j\backslash k}}\big) - \sum_{j:j\neq k} \Psi(\eta_0, \lambda_{j\backslash k})\big\}$, where $\lambda_{j\backslash k} = \mathbf{y}^T\mathbf{P}_{\bar{S}_j}\mathbf{y}/\sigma^2$ when $\mathbf{x} \in \bar{S}_k$. This bound depends on η_0 and, as shown in Sec. 5, is relatively loose for the advertised value of $\eta_0 = 0.25$ in [18].

4. DISCUSSION

This section provides interpretations of Theorems 1 and 2 in terms of invariance properties of the tests and the influence of interference and the geometry between subspaces on performance measures.

Invariance properties: Examining the test statistics for both signal and active subspace detection in Theorem 1, one can see that the test statistics are invariant to the rotations and translations in T. Moreover, any rotations of $\bar{S}_{\widehat{k}}$ do not change the energy in the observed signal and thus the statistics are invariant to such rotations. Finally, the detector is also invariant to any scaling of the observed signal.

Effect of geometry between interference-free subspaces: The performance for signal detection slightly decreases as the angles between the interference-free subspaces increase. The intuition for this lies in the fact that, for signal detection, confusing a signal coming from one subspace as a signal from another subspace is not important as long as a signal of interest is present. When the interference-free subspaces are far apart, the chance of this confusion is lower and thus the detection performance decreases slightly.

For active subspace detection, however, the confusion between interference-free subspaces is crucial and the geometry does have a strong impact, as given by the following theorem.

Theorem 3. *For the active subspace detection test in Theorem 1, the lower bound on the probability of correct classification given in Theorem 2 increases with increasing principal angles between the interference-free subspaces.*

One can prove this theorem by following along the lines of the proof of our earlier paper [19, Theorem 7]. Specifically, one just needs to examine the principapl angles between the interference-free subspaces to complete the proof. Intuitively speaking, the probability of correct classification depends on identifyng the true subspace generating the signal of interet. As the angles between the unaccouted subspaces inrease, the chance of confusing subspaces with each other decreases and the correct classification performance improves.

Effect of geometry of the interference subspace: The effect of interference on performance measures can be seen through $\mathbf{y}^T\mathbf{P}_{\bar{S}_{\widehat{k}}T}^{\perp}\mathbf{y}$ in the denominator of the proposed test statistics. Both detection tests in Theorem 1 are directly proportional to $\arg\min_k \mathbf{y}^T\mathbf{P}_{\bar{S}_kT}^{\perp}\mathbf{y}$, and thus directly proportional to $\arg\max_k \|\mathbf{P}_{\bar{S}_{\widehat{k}}}\mathbf{P}_T^{\perp}\mathbf{y}\|^2$ (since $\mathbf{y}^T\mathbf{P}_{\bar{S}_kT}^{\perp}\mathbf{y} = \mathbf{I} - \mathbf{P}_T - \mathbf{y}^T\mathbf{P}_T^{\perp}\mathbf{P}_{\bar{S}_k}\mathbf{P}_T^{\perp}\mathbf{y}$). This means that as the subspaces in the union move away from the interference subspace, and closer to the subspace orthogonal to the interference ($T^{\perp}$), the detection performance of both tests improves.

5. NUMERICAL EXPERIMENTS

In this section we provide numerical experiments conducted on synthetic data to characterize the performance of the proposed tests and validate the observations made in Sec. 4. To this end, we run Monte-Carlo experiments as follows: we generate 2.5×10^3 instances of three 2-dimensional subspaces

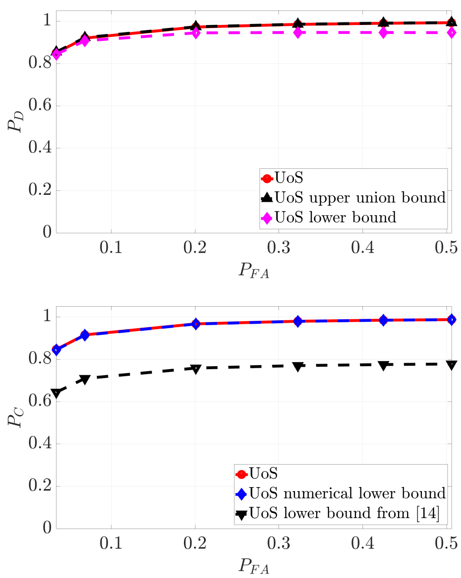

Fig. 1. ROC curves for signal (top) and active subspace (bottom) detection along with their respective bounds.

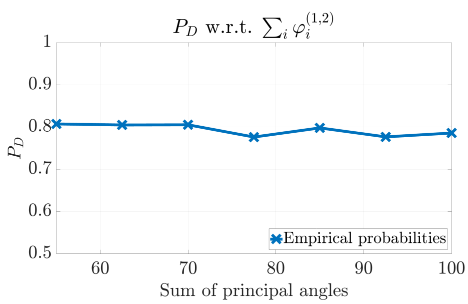

Fig. 2. Signal detection probability as a function of sum of principal angles between interference-free subspaces.

in a 6-dimensional space. For each instance, we consider one subspace to be the interference subspace and the other two subspaces comprise the union. Then we run 10^5 trials for each instance and use the proposed tests to detect signals and the active subspace, with the threshold computed numerically for each false alarm rate. The receiver operating characteristic (ROC) curves for signal and active subspace detection with their respective bounds can be seen in Fig. 1 for a signal to interference and noise ratio (SINR) of 20dB. It can be seen from the figure that the bounds derived in this paper are very close to the actual probabilities of detection for high SINR.

Next, the effect of angles between interference-free subspaces on the probabilities of signal and active subspace detection is highlighted in Fig. 2 and Fig. 3, respectively. Plots are shown for SINR of 10dB for only one subspace and with respect to the sum of angles between interference-free subspaces. Similar behavior is observed for the other subspace and for the individual angles as well. As noted in Sec 4, the pro-

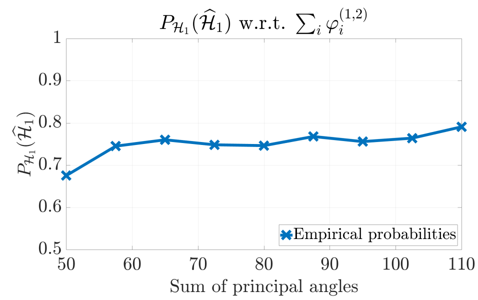

Fig. 3. Active subspace detection probability as a function of sum of principal angles between interference-free subspaces.

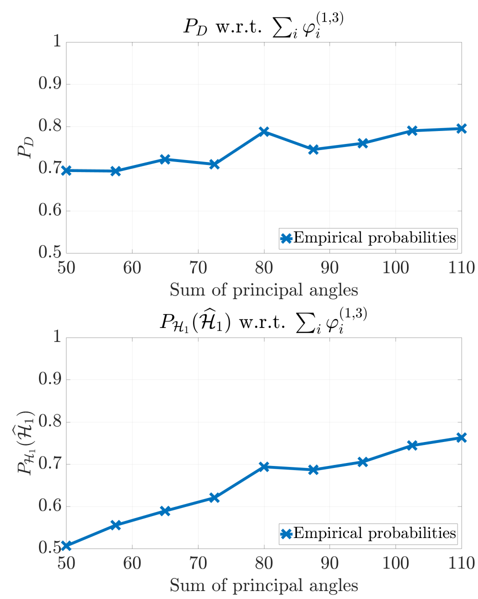

Fig. 4. Signal (top) and active subspace (bottom) detection probabilities as a function of sum of principal angles between a subspace in the union and the interference subspace.

bability of signal detection slightly decreases while the probability of correct classification increases with the increasing principal angles φ_i between interference-free subspaces. Finally, one can see from Fig. 4 (SINR 10dB) (plots shown for one subspace and for sum of angles) that as the subspaces in the union move away from the interference subspace, both the probabilities of signal and active subspace detection improve significantly.

6. CONCLUSION

We derived statistical tests for the signal and active subspace detection problems under the UoS model with interference. We also obtained performance bounds for the derived tests and analyzed these bounds in light of the geometry between different subspaces. Finally, we verified the various observations made in the paper through Monte-Carlo simulations.

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