TARGET ESTIMATION IN COLOCATED MIMO RADAR VIA MATRIX COMPLETION

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ABSTRACT

We consider a colocated MIMO radar scenario, in which the receive antennas forward their measurements to a fusion center. Based on the received data, the fusion center formulates a matrix which is then used for target parameter estimation. When the receive antennas sample the target returns at Nyquist rate, and assuming that there are more receive antennas than targets, the data matrix at the fusion center is low-rank. When each receive antenna sends to the fusion center only a small number of samples, along with the sample index, the receive data matrix has missing elements, corresponding to the samples that were not forwarded. Under certain conditions, matrix completion techniques can be applied to recover the full receive data matrix, which can then be used in conjunction with array processing techniques, e.g., MUSIC, to obtain target information. Numerical results indicate that good target recovery can be achieved with occupancy of the receive data matrix as low as 50%.

Index Terms— Array processing, compressed sensing, matrix completion, MIMO radar, MUSIC

1. INTRODUCTION

Multiple-input and multiple-output (MIMO) radar systems have received considerable attention in recent years due to their superior target estimation performance. Colocated MIMO radar systems exploit waveform diversity to formulate a long virtual array with number of elements equal to the product of the number of transmit and receive antennas. As a result, they achieve higher resolution than traditional phased array radars for the same amount of data [1][2]. Compressed sensing (CS) enables MIMO radar systems to maintain their advantages while significantly reducing the required measurements per receive antenna [3][4]. In CS-based MIMO radar, target parameters are estimated by exploiting the sparsity of targets in the angle, Doppler and range space, referred to as the *target space*. For CS-based sparse target estimation, the target space needs to be discretized into a fine grid, based on which the CS sensing matrix is constructed. However, performance of CS-based MIMO radar degrades when targets

fall between grid points, a case also known as basis mismatch [5] in the CS literature.

Another approach related in spirit to CS is that of matrix completion. Matrix completion aims to recover a low-rank data matrix from partial samples of its entries by solving a relaxed nuclear norm optimization problem [6][7]. Array signal processing with matrix completion has been studied in [8][9]. To the best of our knowledge, however, matrix completion has not been exploited for target estimation in colocated MIMO radar. Our paper is related to the ideas in [9] in the sense that matrix completion is applied to the received data matrix formed by an array. However, due to the unique structure of the received signal in MIMO radar, the problem formulation and treatment in here is different than that in [9].

The main idea of our work is as follows. We consider a colocated MIMO radar scenario in which receive antennas forward their measurements to a fusion center. Based on the received data, the fusion center formulates a matrix, which is then used for estimating the target parameters. When the receive antennas sample the target returns at Nyquist rate, and assuming that there are more receive antennas than targets, the data matrix at the fusion center is low-rank. When each receive antenna sends to the fusion center only a small number of samples, along with the sample index, the receive data matrix has missing elements, corresponding to the samples that were not forwarded. Under certain conditions, matrix completion techniques can be applied to recover the full receive data matrix, which can then be used in conjunction with parametric methods such as MUSIC to obtain target information. Compared to CS MIMO radar, our proposed method has the same advantage in terms of reduction of samples needed for accurate estimation but it avoids the basis mismatch issue inherent in CS-based approach.

The rest of the paper is organized as follows. The colocated MIMO radar system model is described in Section 2. Some background of noisy matrix completion is introduced in Section 3. The applicability of matrix completion to MIMO radar is discussed in Section 4, and numerical results are given in Section 5. Finally, Section 6 provides concluding remarks.

2. COLOCATED MIMO RADAR SYSTEM

We consider a MIMO radar system that employs colocated transmit and receive antennas, shown in Fig. 1. We use
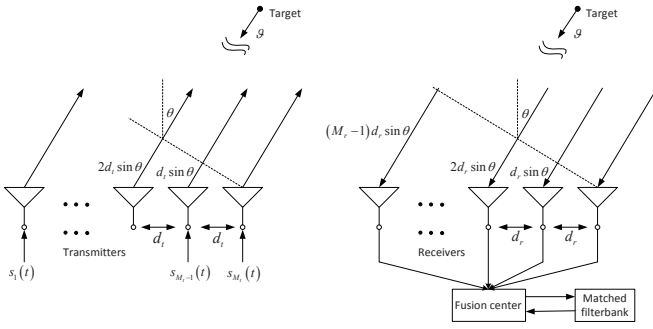
Fig. 1. Colocated MIMO radar system under the ULA model.

M_t and M_r to denote the numbers of transmit antennas and receive antennas, respectively. Although our results can be extended to an arbitrary antenna configuration, the results here are presented for the case in which the transmit and receive antennas form a uniform linear array (ULA) with inter-element spacing between transmit and receive antennas d_t and d_r, respectively. We further assume $d_t = d_r = \lambda/2$, where λ is the wavelength of the carrier signal. Further, the waveforms $s_k(t), k = 1, ..., M_t$, transmitted from the transmit antennas are assumed to be narrow-band and orthogonal. Now suppose there are K point targets in the far field at angles $\theta_k, k = 1, ..., K$, each moving with speed ϑ_k. The corresponding $M_t \times K$ dimensional transmit steering matrix can be expressed as $\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), ..., \mathbf{a}(\theta_K)]$, where

$$\mathbf{a}(\theta_k) = \left[1, e^{j\frac{2\pi}{\lambda}d_t \sin(\theta_k)}, ..., e^{j\frac{2\pi}{\lambda}(M_t-1)d_t \sin(\theta_k)}\right]^T, \quad (1)$$

and the $M_r \times K$ dimensional receive steering matrix $\mathbf{B}(\theta)$ can be defined in a similar fashion based on the receive steering vectors $\mathbf{b}(\theta_k)$.

In order to estimate the speed of each target, multiple pulses need to be transmitted. Let Q be the number of transmitted pulses, with T_{PRI} being the pulse repetition interval. Assume the target reflection coefficients $\{\beta_k\}, k = 1, ..., K$ are complex and remain constant during the Q pulses. For slowly moving targets, $(2\vartheta T_p/\lambda \ll 1$, where T_p is the pulse duration) the Doppler shift within a pulse can be ignored, while the Doppler changes from pulse to pulse.

Under the narrowband assumption the received signal at the lth receive antenna can be approximated as [4]

$$x_l(t) \approx \sum_{k=1}^{K} \beta_k e^{j\frac{2\pi}{\lambda}2\vartheta_k t} b_l(\theta_k) \mathbf{a}^T(\theta_k)\mathbf{s}(t) + w_l(t). \quad (2)$$

Here $\mathbf{s}(t) = [s_1(t), ..., s_{M_t}(t)]^T$.

Suppose that each receive antenna samples the received signal at rate L/T_s and forwards the samples to the fusion center. Here, T_s is the sampling time, L is the number of nonzero samples and we assume $L \gg K$. At the fusion cen-

ter, received data during the qth pulse can be written as [10]

$$\mathbf{X}_q = \mathbf{B}(\theta)\mathbf{\Sigma}\mathbf{D}_q\mathbf{A}^T(\theta)\mathbf{S} + \mathbf{W}_q = \mathbf{Z}_q + \mathbf{W}_q, \quad (3)$$

where $\mathbf{\Sigma} = \text{diag}([\beta_1, ..., \beta_K])$; $\mathbf{D}_q = \text{diag}(\mathbf{d}_q)$, with $\mathbf{d}_q = \left[e^{j2\pi 2\vartheta_1(q-1)T_{PRI}}, ..., e^{j2\pi 2\vartheta_K(q-1)T_{PRI}}\right]^T$; $\mathbf{S} = [\mathbf{s}(0T_s), ..., \mathbf{s}((L-1)T_s)]$; and $\mathbf{W}_q$ is a Gaussian noise matrix. Note that both matrices $\mathbf{\Sigma}$ and $\mathbf{D}_q$ are rank-K, while the rank of matrix $\mathbf{S}$ is $\min\{M_t, L\}$. Thus, for $M_t > K$ the rank of the noise free data matrix $\mathbf{Z}_q = \mathbf{B}(\theta)\mathbf{\Sigma}\mathbf{D}_q\mathbf{A}^T(\theta)\mathbf{S}$ is K. In other words, the data matrix $\mathbf{Z}_q$ is low-rank based on the assumption that $M_r \gg K$.

3. MATRIX COMPLETION WITH NOISE

We now provide a brief overview of the problem of recovering a rank r matrix $\mathbf{M} \in \mathbb{C}^{n_1 \times n_2}$ based on partial knowledge of its entries, possibly corrupted by noise, i.e., $[\mathbf{Y}]_{ij} = [\mathbf{M}]_{ij} + [\mathbf{E}]_{ij}, (i,j) \in \Omega$, where, $[\mathbf{E}]_{ij}$ represents noise and Ω is the set of observed entries. This can also be expressed as $\mathcal{P}_\Omega(\mathbf{Y}) = \mathcal{P}_\Omega(\mathbf{M}) + \mathcal{P}_\Omega(\mathbf{E})$, where $\mathcal{P}_\Omega$ represents the sampling operation. According to [7], when $\mathbf{M}$ is low-rank and its singular vectors are sufficiently spread, i.e., both the numbers of zero and large elements in the singular vectors are not large, $\mathbf{M}$ can be recovered by solving a relaxed nuclear norm optimization problem, given by

$$\min \|\mathbf{X}\|_* \quad \text{s.t.} \quad \|\mathcal{P}_\Omega(\mathbf{X} - \mathbf{Y})\|_F \leq \delta \quad (4)$$

where $\|\cdot\|_*$ denotes the nuclear norm, i.e., the sum of singular values of $\mathbf{X}$, while $\delta > 0$ is a constant.

To test the 'sufficiently spread' requirement, the *strong incoherence property* of matrix $\mathbf{M}$ with parameter μ has been introduced in [6]. Consider the singular value decomposition (SVD) of $\mathbf{M}$, i.e., $\mathbf{M} = \sum_{k=1}^{r} \rho_k \mathbf{u}_k \mathbf{v}_k^H$, and define $P_U = \sum_{1 \leq i \leq r} \mathbf{u}_i \mathbf{u}_i^H$, $P_V = \sum_{1 \leq i \leq r} \mathbf{v}_i \mathbf{v}_i^H$, and $\mathbf{T} = \sum_{1 \leq i \leq r} \mathbf{u}_i \mathbf{v}_i^H$. When the following conditions are satisfied, the matrix $\mathbf{M}$ is said to satisfy the strong incoherence property with $\mu = \max(\mu_1, \mu_2)$.

A1) For all pairs $(a, a') \in [n_1] \times [n_1]$ and $(b, b') \in [n_2] \times [n_2]$, there is $\mu_1 > 0$ such that

$$\left|\langle \mathbf{e}_a, P_U \mathbf{e}_{a'}\rangle - \frac{r}{n_1}1_{a=a'}\right| \leq \mu_1 \frac{\sqrt{r}}{n_1}, \quad (5)$$

$$\left|\langle \mathbf{e}_b, P_V \mathbf{e}_{b'}\rangle - \frac{r}{n_2}1_{b=b'}\right| \leq \mu_1 \frac{\sqrt{r}}{n_2}, \quad (6)$$

where $\mathbf{e}_a$ is the vector with the ath element equal to 1 and others being zero, while $1_{a=a'}$ indicates that it is equal to 1 when $a = a'$ and 0 otherwise.

A2) For all $(a, b) \in [n_1] \times [n_2]$, there exists a constant $\mu_2 > 0$ such that $|T_{ab}| \leq \mu_2 \frac{\sqrt{r}}{\sqrt{n_1 n_2}}$, where T_{ab} is the (a, b) entry of the matrix $\mathbf{T}$.

Note that it has been shown in [6] that when the maximum (in magnitude) values of the K left and right singular vectors are bounded, i.e.,

$$\|\mathbf{u}_k\|_{\ell_\infty} \le \sqrt{\frac{\mu_B}{n_1}}, \|\mathbf{v}_k\|_{\ell_\infty} \le \sqrt{\frac{\mu_B}{n_2}} \tag{7}$$

with $\mu_B = O(1)$, then the strong incoherence property is guaranteed with $\mu \le \mu_B \sqrt{r}$.

Now define $n = \max(n_1, n_2)$ and suppose $\mathbf{M}$ satisfies the strong incoherence property. Then [6] establishes in the noiseless case that, by observing N randomly selected entries with $N \ge C\mu^2 nr\log^6 n$ for some constant C, the matrix $\mathbf{M}$ can be recovered exactly with a probability of at least $1 - n^{-3}$. Further, [7] establishes that, when observations are corrupted with white noise $[\mathbf{E}]_{ij}$ that is zero-mean Gaussian with variance σ^2, the recovery error is bounded as $\left\| \mathbf{M} - \hat{\mathbf{M}} \right\|_F \le 4\sqrt{\frac{1}{p}(2+p)\min(n_1, n_2)}\delta + 2\delta$, where $p = \frac{N}{n_1 n_2}$ is the fraction of observed entries.

4. MATRIX COMPLETION FOR MIMO RADAR

The left singular vectors of $\mathbf{Z}_q$ defined in (3) are the eigenvectors of $\mathbf{Z}_q \mathbf{Z}_q^H = \mathbf{H}\mathbf{S}\mathbf{S}^H \mathbf{H}^H$, where $\mathbf{H} = \mathbf{B}(\theta)\mathbf{\Sigma}\mathbf{D}_q \mathbf{A}^T(\theta)$. The right singular vectors of $\mathbf{Z}_q$ are the eigenvectors of $\mathbf{S}^H \mathbf{H}^H \mathbf{H}\mathbf{S}$. Since $\mathbf{S}$ is orthogonal, it holds that $\mathbf{S}\mathbf{S}^H = \mathbf{I}$. Thus, the left singular vectors are only determined by matrix $\mathbf{H}$, while the right singular vectors are affected by both transmit waveforms and matrix $\mathbf{H}$.

For the problem considered in this paper, it is difficult to determine analytically the behavior of the entries of the left and right singular vectors of $\mathbf{Z}_q$. Instead, we get an idea of the behavior of the maximum values and the parameters that affect their spread using simulations. We consider a MIMO radar setup in which the target direction of arrival (DOA) angles are uniformly distributed in $[-90°, 90°]$ and the corresponding target speeds are uniformly distributed in the range $[150, 450]\,m/s$. In addition, β_k are following complex Gaussian distribution and kept unchanged for $Q = 10$ pulses. The pulse repetition interval is $T_{PRI} = 1/4000$ second, and the carrier frequency is $f = 10^9$ Hz, resulting in $\lambda = c/f = 0.3$ meter. Two types of orthogonal waveforms are considered: Hadamard and Gaussian orthogonal waveforms. Several cases of parameters are verified. Case I: $M_r = 40$, $L = 128$; Case II: $M_r = 1000$, $L = 128$; Case III: $M_r = 40$, $L = 1024$. Each case runs for 300 iterations.

Let m_1 and m_2 denote the maximum element value of K left and right singular vectors of $\mathbf{Z}_q$. The complementary cumulative distribution function (CDF) curves of m_1 and m_2, i.e., $\Pr(m > m_i), i = 1, 2$ are plotted in Fig. 2 for Cases I and II. It can be seen from these plots that as M_r increases, the bounds of m_1 for both $K = 2$ and $K = 10$ decrease, while the distribution of m_2 does not significantly change when L
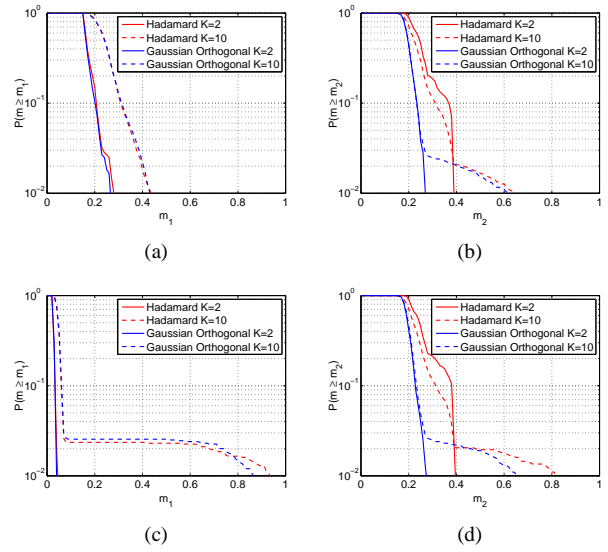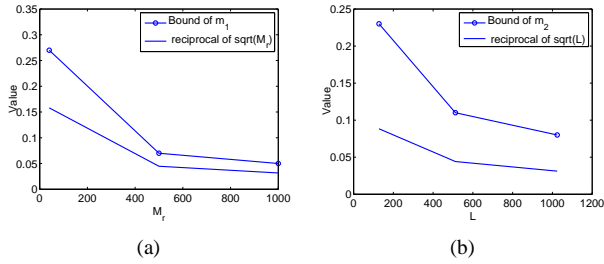

Fig. 2. Complementary CDF of m_1 and m_2. Case I: $M_r = 40$, $L = 128$; (a) left singular vectors, (b) right singular vectors; Case II: $M_r = 1000$, $L = 128$; (c) left singular vectors, (d) right singular vectors.

is fixed. Figure 2 (c) shows that as M_r gets large, m_1 gets bounded by a small number with high probability. Space limitations prevent us from displaying more figures of Case III, which show that m_2 is bounded by a small number with high probability as L gets large.

Extensive simulations also show that the bounds on m_1 and m_2 scale as $1/\sqrt{M_r}$ and $1/\sqrt{L}$, respectively, with some constant $\sqrt{\mu_B}$. For Gaussian orthogonal waveform and $K = 2$, $\mu_B \approx 2.4$ for the bound of m_1 (see Fig. 3 (a)) and $\mu_B \approx 6.5$ for the bound of m_2 (see Fig. 3 (b)). For $K = 10$ and Hadamard waveform, the scaling laws also hold but with larger constants μ_B. Therefore, depending on the number of receive antennas M_r and the number of samples L in one pulse, m_1 and m_2 can be assumed to be bounded by small numbers. Based on [6], therefore, we conclude that the strong incoherence property is likely satisfied in our problem.

It is also worth noting that under Gaussian orthogonal waveforms, m_2 is concentrated in a smaller range as compared with the range for Hadamard waveforms. This indicates that the waveform indeed plays a role for the use of matrix completion, and perhaps the waveform can be optimally designed to result in low probability of large values for m_2.

Since the matrix completion conditions appear to be satisfied in our case, we propose that each antenna obtains and forwards to the fusion center a small number of samples during each pulse. Note that along with the samples, the antenna needs to inform the fusion center on how the sample was obtained so that the fusion center can determine where to position the received samples in the receive data matrix $\mathbf{Z}_q$. In a practical setting, a pseudo-random sampling ADC at each re-

(a)　　　　　　　　　　(b)

Fig. 3. Bounds of m_1 and m_2 scales with reciprocal of $\sqrt{M_r}$ and $\sqrt{L}$, respectively.

(a)　　　　　　　　　　(b)

Fig. 4. Performance comparisons: (a) Relative error of the recovered data matrix; (b) Probability of DOA estimation resolution.

ceive antenna could be used in the place of the Nyquist sampler, where the pseudo-random generator seeds would be distributed to the receive antennas by the fusion center. Finally, the fusion center can recover the data matrix $\hat{\mathbf{X}}_q$ by applying matrix completion to the received data.

Once the data matrix $\hat{\mathbf{X}}_q$ is recovered, it can go through a matched-filter bank to produce

$$\mathbf{Y}_q = \frac{1}{L}\hat{\mathbf{X}}_q \mathbf{S}^H = \mathbf{B}\left(\theta\right)\mathbf{\Sigma}\mathbf{D}_q\mathbf{A}^T\left(\theta\right) + \tilde{\mathbf{W}}_q, \qquad (8)$$

where $\tilde{\mathbf{W}}_q$ is noise whose distribution is a function of the additive noise $\mathbf{W}_q$ and the nuclear norm minimization problem in (4). Next, stacking $\mathbf{Y}_q \in \mathbb{C}^{M_r \times M_t}$ into a vector $\mathbf{y}_q$, and based on the vectors corresponding to Q pulses, the following matrix can be formed: $\mathbf{Y}_R = [\mathbf{y}_1, ..., \mathbf{y}_Q] \in \mathbb{C}^{M_r M_t \times Q}$. Reshaping $\mathbf{Y}_R$ into $\mathbf{Y} \in \mathbb{C}^{Q M_t \times M_r}$, we have

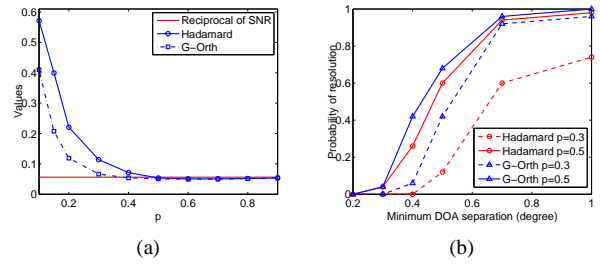$$\mathbf{Y} = \mathbf{F}\mathbf{\Sigma}\left[\mathbf{b}\left(\theta_1\right), ..., \mathbf{b}\left(\theta_K\right)\right] + \mathbf{W}, \qquad (9)$$

where $\mathbf{F} = [\mathbf{d}\left(\vartheta_1\right) \otimes \mathbf{a}\left(\theta_1\right), ..., \mathbf{d}\left(\vartheta_K\right) \otimes \mathbf{a}\left(\theta_K\right)]$, $\mathbf{d}\left(\vartheta\right) = \left[1, e^{j2\pi 2\vartheta T_{PRI}}, ..., e^{j2\pi 2\vartheta(Q-1)T_{PRI}}\right]^T$, with $\otimes$ denoting the Kronecker product. The sampled covariance matrix of the receive data signal can then be obtained as $\hat{\mathbf{R}}_Y = \frac{1}{M_r}\mathbf{Y}\mathbf{Y}^H$, based on which target estimation can be implemented using any array processing method such as MUSIC.

5. NUMERICAL RESULTS

In this section we present some simulation results on the performance of the proposed method. We use the simulation setting considered in the previous section, i.e., $M_t = 20$, $M_r = 40$, $Q = 5$, $L = 128$. The signal-to-noise ratio (SNR) is set to 25 dB, while β_k are following complex Gaussian distribution and kept unchanged for Q pulses. For matrix completion, the TFOCS software package [11] is used.

First, we plot relative errors (averaged over 50 Monte Carlo runs) of the received data matrix $\hat{\mathbf{X}}_q$ for Hadamard and Gaussian orthogonal (G-Orth) waveforms. The relative error is defined as $\frac{\left\|\mathbf{Z}_q - \hat{\mathbf{X}}_q\right\|_F}{\left\|\mathbf{Z}_q\right\|_F}$, where $\mathbf{Z}_q$ is the data matrix calculated without missing elements. Under each waveform,

$K = 2$ point targets in the far field are randomly generated. The result is shown in Fig. 4 (a) for $q = 1$. It can be seen from this figure that, as p increases, the relative recovery error of the data matrix under Gaussian orthogonal waveform reduces to the reciprocal of the SNR faster than that under Hadmard waveform. A plausible reason for this is that under the Gaussian orthogonal waveform, the maximum value of elements in the singular vectors of $\mathbf{Z}_q$ is bounded by a smaller number with high probability, as compared with that under the Hadamard waveform (see Fig. 2).

Next, the probabilities of DOA estimation resolution under the two orthogonal waveforms are plotted in Fig. 4 (b) for the following scenario. Two targets are randomly generated among DOA range $[-15°, 15°]$ with minimum DOA separations $d_\theta = [0.2°, 0.3°, 0.4°, 0.5°, 0.7°, 1°]$ and the corresponding speeds are set to 150 and 400 m/s. The MUSIC algorithm is applied to obtain the target DOA information. If the DOA estimates $\hat{\theta}_i$, $i = 1, 2$ satisfy $\left|\theta_i - \hat{\theta}_i\right| \leq \varepsilon d_\theta, \varepsilon = 0.1$, we declare this as a success. The probability of DOA resolution is then defined as the fraction of successful events in 50 iterations. It can be seen from the figure that when $p = 0.3$, the Gaussian orthogonal waveform has a much better DOA estimation resolution compared with Hadamard waveform. As p increases to 0.5, the performance difference becomes small since the relative recovery errors under both waveforms are similar (see Fig. 4 (a)). Figure 4 confirms that Gaussian orthogonal waveforms are better than Hadamard waveforms for matrix completion-based DOA estimation.

6. CONCLUSIONS

We have provided results suggesting that matrix completion can be used in MIMO radar to reduce the number of data needed to be communicated to the fusion center by each receive antenna. Numerical results show that matrix completion in conjunction with MUSIC can achieve accurate target estimation with sub-Nyquist samples. Thus, the proposed method can result in significant savings in terms of data that need to be obtained at the receive antennas and subsequently transmitted to the fusion center.

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